Int. J. Heat Mass Transfer. Vol. 13, pp. 1231–1233. Pergamon Press 1970. Printed in Great Britain

SHORTER COMMUNICATIONS

PHASE CHANGE OF SPHERICAL BODIES

SUNG HWAN CHO* and J. EDWARD SUNDERLAND

Department of Mechanical and Aerospace Engineering, North Carolina State University, Raleigh, North Carolina, U.S.A.

(*Received* 29 *September* 1969 *and in revised form* 22 *December* 1969)

1. INTRODUCTION

HEAT conduction problems involving phase changes have been studied extensively since Stefan [1] published his analytical solution. There are, however, only a few investigations [2–6] reported for phase changes involving spherical bodies.

In this paper, both inward and outward phase changes of spherical bodies are considered. The initial temperature of the system is assumed constant at the fusion temperature and the boundary surface temperature is assumed to change instantaneously. A simple approximation is obtained and a numerical solution using a finite difference method is employed to test the accuracy of the approximation.

2. ANALYSIS

Consider a sphere with radius $r = a$, or a liquid space bounded internally by a sphere of radius a. The temperature T of the system is initially at the fusion temperature T_f. At time $t = 0$, the surface temperature is changed and held constant at T_0. The liquid freezes from the surface in either an inward or outward direction.

The temperature distribution in the solid region satisfies the following equations:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial \tau}, \qquad 0 < x < S(\tau), \qquad \tau > 0 \tag{1}$$

$$u = 0 \quad \text{at} \quad \tau = 0 \qquad \text{or at} \quad x = S(\tau), \qquad \tau > 0 \tag{2}$$

$$u = 1 \quad \text{at} \quad x = 0, \qquad \tau > 0 \tag{3}$$

$$\frac{\partial u}{\partial x} = -L(1 + p)\frac{\mathrm{d}S}{\mathrm{d}\tau} \quad \text{at} \quad x = S(\tau), \quad \tau > 0 \tag{4}$$

$$S(0) = 0. \tag{5}$$

The dimensionless variables are defined by

$$u = \frac{r}{a}\frac{T - T_f}{T_0 - T_f} \tag{6}$$

$$x = \frac{a - r}{a} \text{ for inward freezing} \tag{7a}$$

$$x = \frac{r - a}{a} \text{ for outward freezing} \tag{7b}$$

$$\tau = \frac{\alpha t}{a^2} \tag{8}$$

$$S(\tau) = \frac{a - R(t)}{a} \text{ for inward freezing} \tag{9a}$$

$$S(\tau) = \frac{R(t) - a}{a} \text{ for outward freezing} \tag{9b}$$

$$L = \frac{H}{c_p(T_f - T_0)} \tag{10}$$

where α is the thermal diffusivity, c_p is the specific heat, and $R(t)$ is the position of the phase front. The enthalpy of fusion, H, is positive for freezing and negative for melting so that the dimensionless variable L is always positive. All properties are assumed to be independent of temperature. In equation (4), $p(\tau)$ is defined by

$$p(\tau) = -S(\tau) \text{ for inward phase change, and} \tag{11}$$

$$p(\tau) = S(\tau) \text{ for outward phase change.} \tag{12}$$

Note that equations (1)–(5) become the classical Stefan problem when $p(\tau) = 0$ and $u = (T - T_f)/(T_0 - T_f)$. Therefore, for a flat plate, $p(\tau) = 0$.

3. APPROXIMATE SOLUTION

Assume the temperature distribution, $u(x, \tau)$ to be similar to the exact solution for the one-dimensional Stefan problem. Then

$$u(x, \tau) = 1 - \frac{\operatorname{erf}[\lambda x/S(\tau)]}{\operatorname{erf}\lambda}, \quad \tau > 0, \quad 0 < x < S(\tau). \tag{13}$$

Here λ is considered a function of time.

Substituting equation (13) into equation (4), one obtains

$$S(1 + p)\frac{\mathrm{d}S}{\mathrm{d}\tau} = \frac{2\lambda}{(\sqrt{\pi})\, L\, \mathrm{e}^{\lambda^2} \operatorname{erf}\lambda}. \tag{14}$$

* Current Address: School of Engineering, Tuskegee Institute, Alabama 36088.

For small time, the spherical body acts like a flat plate; therefore, $\lambda \approx \lambda_0$, where λ_0 is determined from Stefan's solution and is the positive solution of

$$(\sqrt{\pi})\,\lambda_0\, e^{\lambda_0^2} \operatorname{erf} \lambda_0 = \frac{1}{L}. \tag{15}$$

Integrating equation (14) assuming $\lambda = \lambda_0$ yields

$$S\sqrt{(1 + \tfrac{2}{3}p)} = 2\lambda_0\sqrt{\tau}. \tag{16}$$

Note that equation (16) is an exact solution to equation (14) for a flat plate where $p \equiv 0$ and $S = 2\lambda_0\sqrt{\tau}$. Therefore

$$f = \sqrt{(1 + \tfrac{2}{3}p)} \tag{17}$$

can be considered the "approximate correction factor" for the spherical geometry. Since $\lambda \approx \lambda_0$ is valid only for very small time, this approximation is valid only for small time.

In order to check the accuracy of the approximation, this problem is solved numerically using a finite difference method.

4. FINITE DIFFERENCE SOLUTION

The following changes in variables are made in order to establish a coordinate system that would be stationary with respect to the moving fusion front:

$$y = \frac{x}{S(\tau)} \tag{18}$$

$$\psi(\tau) = [S(\tau)]^2. \tag{19}$$

Then equations (1)–(5) become

$$\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\frac{d\psi}{d\tau}\, y\frac{\partial u}{\partial y} = \psi\frac{\partial u}{\partial \tau}, \qquad 0 < y < 1, \qquad \tau > 0 \tag{20}$$

$$u = 0 \quad \text{at} \quad y = 1, \qquad \tau > 0 \tag{21}$$

$$u = 1 \quad \text{at} \quad y = 0, \qquad \tau > 0 \tag{22}$$

$$\frac{\partial u}{\partial y} = -\frac{L}{2}(1 + p)\frac{d\psi}{d\tau} \quad \text{at} \quad y = 1, \qquad \tau > 0 \tag{23}$$

$$\psi(0) = 0. \tag{24}$$

The space between $y = 0$ and $y = 1$ is divided into n equal dubdivisions. Equations (20)–(24) can now be formulated in finite difference equations. The following third order polynomial approximation is used for $(\partial u/\partial y)_{y=1}$:

$$\left.\frac{\partial u}{\partial y}\right|_{y=1} \approx \frac{9u_{n-1}^j - 18u_n^j - 2u_{n-2}^j}{6(\Delta y)}. \tag{25}$$

The subscripts indicate the space increments and the superscripts indicate the time increments. From equation (23)

$$\psi^{j+1} = \psi^j - \frac{\Delta\tau}{3(\Delta y)}\frac{1}{L(1 + p)}\{9u_{n-1}^j - 18u_n^j - 2u_{n-2}^j\} \tag{26}$$

for $j = 0, 1, 2, \ldots$. From equation (21)

$$u_i^{j+1} = u_i^j + \frac{1}{\psi^j}\left[\frac{\Delta\tau}{(\Delta y)^2}\{u_{i+1}^j + u_{i-1}^j - 2u_i^j\} + \frac{\Delta\psi^j}{4(\Delta y)}y_i\{u_{i+1}^j - u_{i-1}^j\}\right] \tag{27}$$

for $i = 2, 3, \ldots, n$. From equations (21) and (22),

$$u_1^j = 1 \quad \text{for} \quad j = 0, 1, 2, \ldots \tag{28}$$

$$u_{n+1}^j = 0 \quad \text{for} \quad j = 0, 1, 2, \ldots \tag{29}$$

For the stability of the solution, time increments are restricted approximately (see [6]) by

$$\Delta\tau < \tfrac{1}{5}\,\psi(\Delta y)^2. \tag{30}$$

5. DISCUSSION OF RESULTS

In order to start the numerical calculation, initial approximations for the nondimensional variables u_i^0 and ψ^0 are needed. The exact solution of Stefan's problem for a flat plate is used for the initial approximation and yields:

$$\psi^0 = 0.01 \quad \text{or} \quad S^0 = 0.1 \tag{31}$$

$$\tau^0 = \psi^0/(4\lambda_0^2) \tag{32}$$

$$u_i^0 = 1 - \frac{\operatorname{erf}(\lambda_0 y_i)}{\operatorname{erf}\lambda_0}. \tag{33}$$

Equations (26) and (27) are calculated by setting $n = 10$ and letting $L = 0.01$, 0.2, 1.0, 5 and 100. An IBM 360 digital computer was used for these calculations. The results are shown in Fig. 1.

The finite difference solution is checked with the exact solution for a flat plate by setting $p(\tau) = 0$. Since the initial approximation in this case is also exact, the possible inaccuracies caused by the initial approximation cannot be checked by this method. However, the effect of the initial approximation is checked by making $S^\circ = 0.01$ (or $\psi^\circ = 10^{-4}$). The errors resulting from the initial approximation were shown to be negligible.

From Fig. 1, it can be seen that when L is very large, the numerical solutions approach the approximate solution (16), and when L is very small the numerical solutions approach Stefan's solution. Therefore, these two solutions can be considered the limiting solutions of the problem. It must be noted, however, from equation (15), that the value of λ_0 becomes either zero or infinite when L is infinite or zero, respectively. There is no steady state as discussed in [6], when the initial temperature is equal to the fusion temperature.

The accuracy of the approximation (16) is good for large values of L. Also the accuracy is better for the outward phase change than the inward phase change for the same value of L. This is anticipated since the change of λ is slower for the outward phase change.

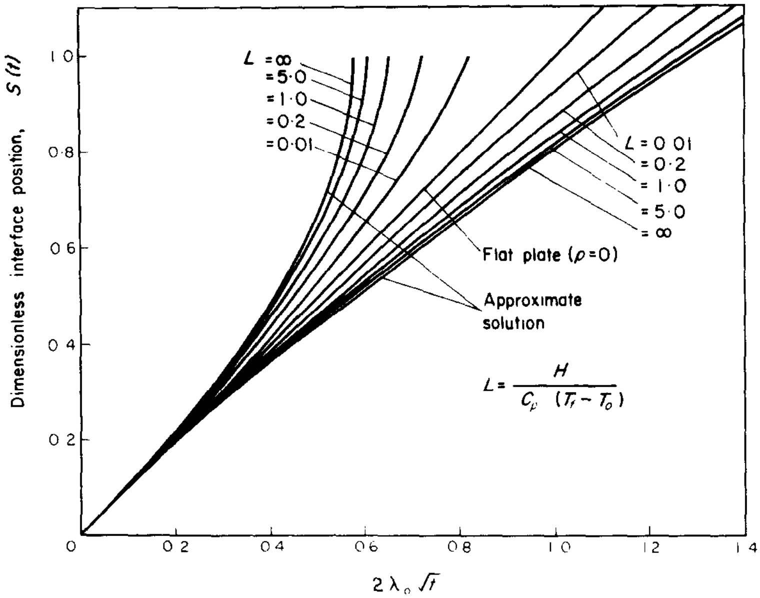

FIG. 1. Interface position as a function of time.

Since λ increases for the inward phase change problem, the approximation always shows a faster rate of phase change for this case. The reverse is true for the outward phase change problem.

6. CONCLUSIONS

A simple approximate solution for the inward or outward phase change of a spherical body is obtained. A finite difference formulation is derived and evaluated to check the accuracy of the approximation.

The accuracy of the approximation is good when the ratio, L, of enthalpy of phase change to the sensible heat, $c_p(T_f - T_0)$ is large. Although many engineering applications would involve problems with large values of L, one application that would satisfy this condition particularly well would the freeze-drying of foods. The approximation has better accuracy for the outward phase change than the inward phase change.

Since the numerical solutions lie between the approximation and the solution of the phase change for a flat plate, the approximation is considered exact as L approaches infinity.

ACKNOWLEDGEMENT

The writers gratefully acknowledge partial support for this investigation by Public Health Service Research Grant FD-00156-03 from the Food and Drug Administration.

REFERENCES

1. J. STEFAN, On the theory of ice formation, especially on ice formation in polar seas, *Ann. Phys.* **42**, 269–286 (1891).
2. F. C. FRANK, Radially symmetric phase growth controlled by diffusion, *Proc. R. Soc.* **A201**, 586–599 (1950).
3. F. KREITH and F. E. ROMIE, A study of the thermal diffusion equation with boundary conditions corresponding to solidification or melting of materials initially at the fusion temperature, *Proc. Phys. Soc.* **68B**, 277–291 (1955).
4. G. A. MARTYNOV, Solution of the inverse Stefan problem in the case of spherical symmetry, *Soviet Phys. Tech. Phys.* **5**, 215–218 (1960).
5. D. LANGFORD, A closed-form solution for the constant velocity solidification of spheres initially at the fusion temperature, *Br. J. Appl. Phys.* **17**, 286 (1966).
6. A. S. TELLER and S. W. CHURCHILL, Freezing outside a sphere, *Chem. Engng Prog. Symp. Series No.*. 59, **61**, 185–189 (1965).